Cartesian Product

$a \neq b$

- If A and B are two sets, then the **Cartesian product** of A and B, written $A \times B$, is the set

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

$(a, b) \neq (b, a)$

A B B A

- (a, b) is called an ordered pair.

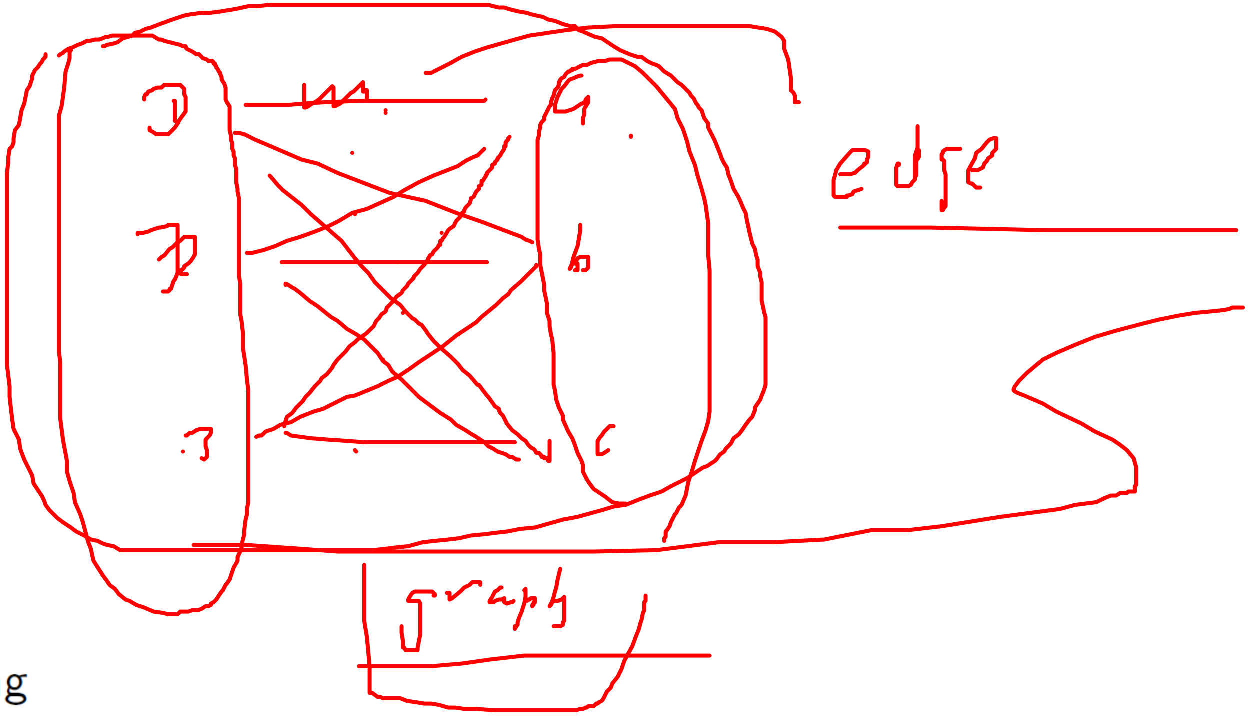
  Unlike with sets, order matters. (a, b) is not the same as (b, a).

$A = B$

$A = \{a, b, c\} = B$

Cartesian products: example

- $A = \{1, 2, 3\}$, $B = \{a, b, c\}$.
- $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$.
- **Note:** $(a, 1)$ is not in $A \times B$. The first thing in the ordered pair has to come from A and the second thing has to come from B.
- **Note:** $|A \times B| = |A| \cdot |B|$.

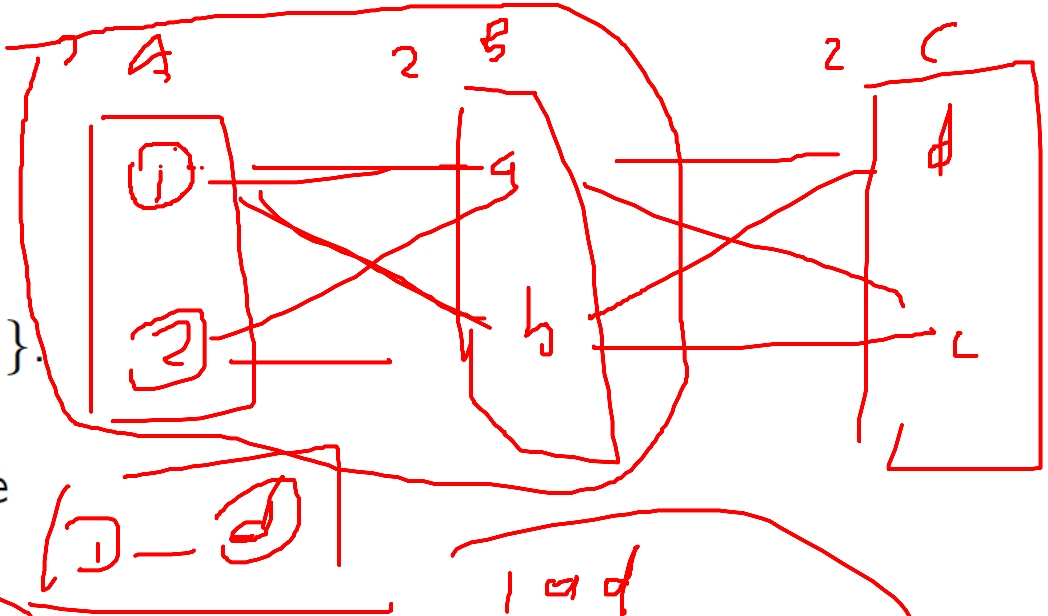


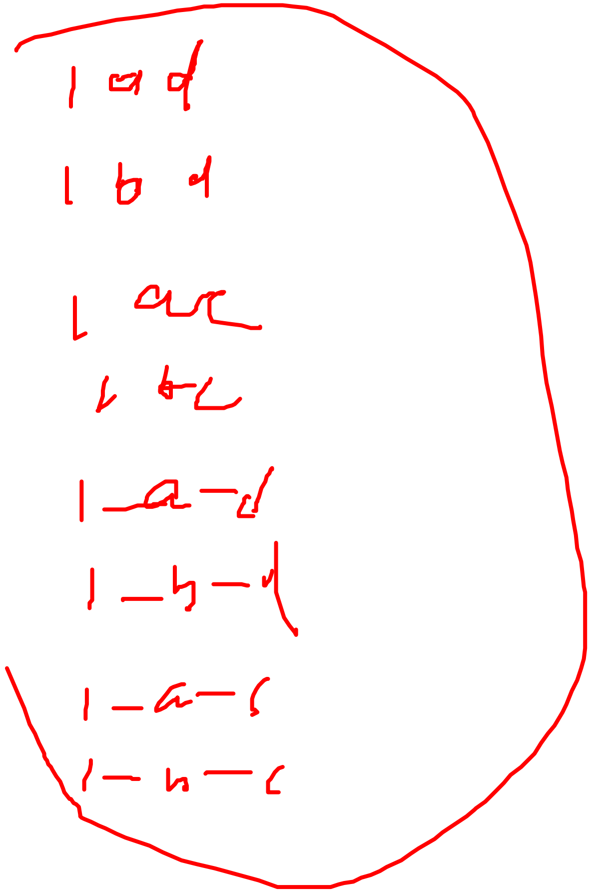
$3 \times 3 = 9$

More on Cartesian products

- The notion of Cartesian product extends to more than two sets:
  - $A \times B \times C = \{(a, b, c) | a \in A, b \in B, c \in C\}$.
    (ordered triples)
  - $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) | a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n\}$.
    (ordered n-tuples)

- **Example:** 3-tuple $2\times2\times2=8$
  - $A = \{1,2\}$, $B = \{a,b\}$, $C = \{\text{dog}, \text{cat}\}$.
  - $A \times B \times C = \{(1,a,\text{dog}), (1,a,\text{cat}), (1,b,\text{dog}), (1,b,\text{cat}), (2,a,\text{dog}), (2,a,\text{cat}), (2,b,\text{dog}), (2,b,\text{cat})\}$.
  - $(A \times B) \times C$ is not the same.
  - $A \times B$ is the set $\{(1,a),(1,b),(2,a),(2,b)\}$, so if we take the Cartesian product of that new set with C, we get
    $\{((1,a),\text{dog}), ((1,a),\text{cat}), ((1,b),\text{dog}), ((1,b),\text{cat}), ((2,a),\text{dog}), ((2,a),\text{cat}), ((2,b),\text{dog}), ((2,b),\text{cat})\}$.

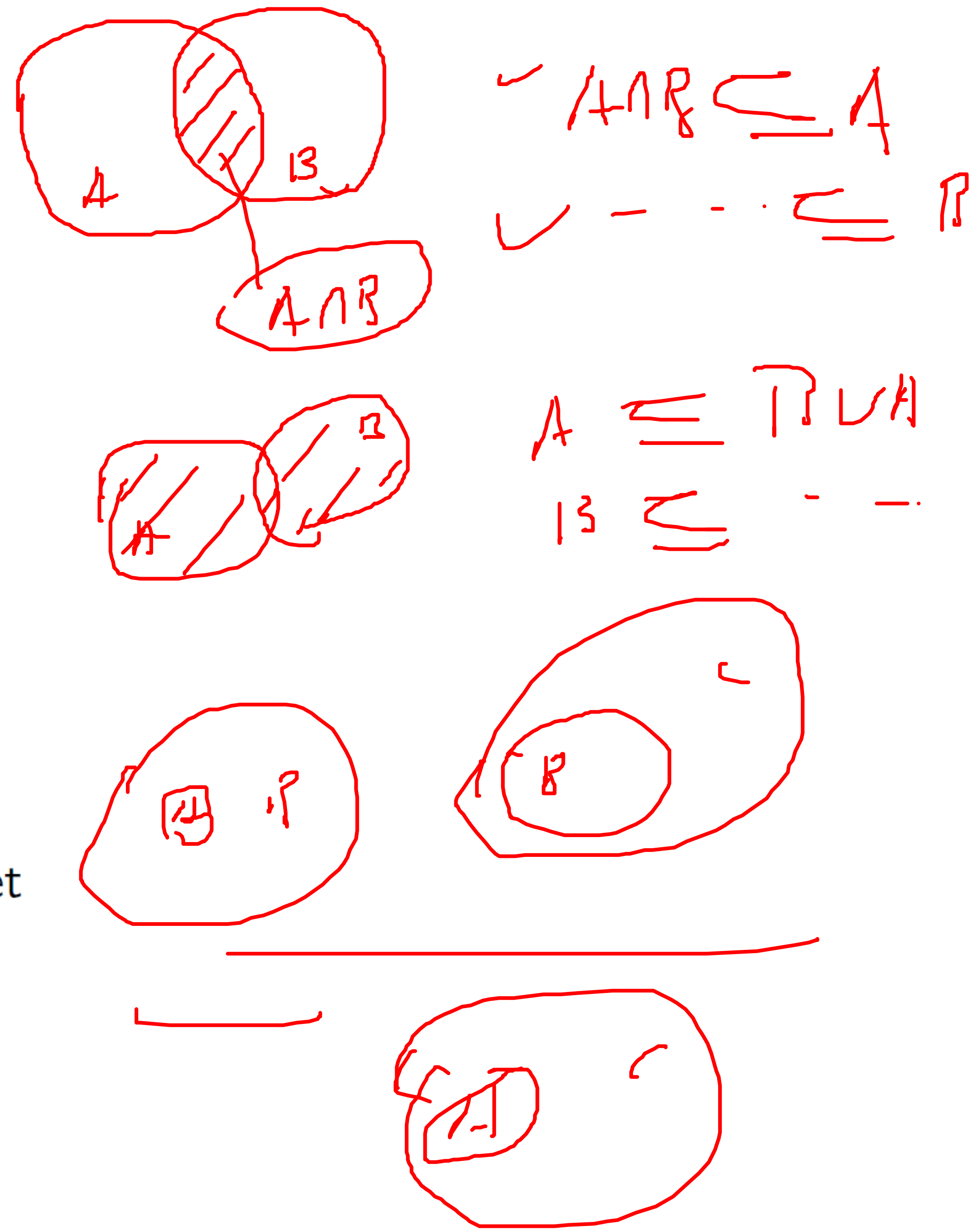




Properties of Sets

Some results discussed in the book

- For all sets in A and B,
  - $\boxed{A \cap B \subseteq A}$ $A \cap B \subseteq A$
  - $A \subseteq B \cup A$ $B \subseteq B \cup A$
- For all sets in A, B and C,
  - $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. $\emptyset$
- Probably the easiest way to convince **yourself** of these things: Venn diagrams.
- But how do we prove **formally** that one set is a subset of another...?



Proofs about subsets

- Think back to the definition of subset.
  - $X \subseteq Y \Leftrightarrow (\forall x, \text{if } x \in X, \text{then } x \in Y)$.

$\forall x.\ x \in X \longrightarrow x \in Y$

- We can use the **element argument** from Section 6.1 to prove general results like these.

$X \subseteq Y$

$X = X \cap Y$

Element Argument:

- Given sets X and Y, to prove that $X \subseteq Y$.

1. Suppose that x is any arbitrary element of X.
2. Prove that x must also be an element of Y.

$X \subseteq X \cup Y$

Example subset proof (element argument)

- For all sets in A and B, $A \cap B \subseteq A$.
- **Proof**: (on the chalkboard).
- This was a very simple example. We will see some more complex subset proofs later.

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to \mathbf{c}$ $\therefore p$

$A \cap B = \{ x \in U \mid x \in A \wedge x \in B \}$

$A = \{ x \in U \mid x \in A \}$

$\boxed{A \cap B \subseteq A} \leftrightarrow \forall x \in U.\ x \in A \cap B \to x \in A$

$\forall x \in U,\ [x \in A \wedge x \in B]$

$\forall x \in U.\ x \in A$

$x \in A$

Proof: $\boxed{\forall x.\ x \in A \cap B}$

$x \in A \to$ true, $\to$ false

$x \in B$

Procedural Versions of Set Definitions

- Let X and Y be subsets of a universal set U and suppose x and y are elements of U.
  - $x \in X \cup Y \Leftrightarrow x \in X$ or $x \in Y$.
  - $x \in X \cap Y \Leftrightarrow x \in X$ and $x \in Y$.
  - $x \in X - Y \Leftrightarrow x \in X$ and $x \notin Y$.
  - $x \in X^c \Leftrightarrow x \notin X$.
  - $(x, y) \in X \times Y \Leftrightarrow x \in X$ and $y \in Y$.

$X \subseteq X \cup Y$

$X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$

$X \cap Y \cap Z$

$X \cup Y \cup Z$

$X_1 \cap X_2 \cap X_3 \cap \cdots X_n$

Set identities

$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

- Many set identities are given in the textbook...
- Let all sets below be subsets of some universal set U.

1. Commutative Laws: For all sets A and B,
   (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.
2. Associative Laws: For all sets A, B and C,
   (a) $(A \cup B) \cup C = A \cup (B \cup C)$ and
   (b) $(A \cap B) \cap C = A \cap (B \cap C)$.
3. Distributive Laws: For all sets A, B and C,
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. Identity Laws: For all sets A,
   (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$.
5. Complement Laws:
   (a) $A \cup A^c = U$ and (b) $A \cap A^c = \emptyset$.
6. Double Complement Law: For all sets A,
   $$(A^c)^c = A.$$
7. Idempotent Laws: For all sets A,
   (a) $A \cup A = A$ and (b) $A \cap A = A$.
8. Universal Bound Laws: For all sets A,
   (a) $A \cup U = U$ and (b) $A \cap \emptyset = \emptyset$.
9. DeMorgan's Laws: For all sets A and B,
   (a) $(A \cup B)^c = A^c \cap B^c$ and (b) $(A \cap B)^c = A^c \cup B^c$.
10. Absorption Laws: For all sets A and B,
    (a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$.
11. Complements of U and $\emptyset$:
    (a) $U^c = \emptyset$ and (b) $\emptyset^c = U$.
12. Set Difference Law: For all sets A and B,
    $$A - B = A \cap B^c.$$

$\emptyset$

- As with our logical equivalences and rule inferences: our concern is not that you be able to memorize all of these, but that you understand them and be able to use them.
- We will also look at how to **prove** statements like these...

$$\underbrace{A \cap (B \cup C)}_{S_1} = \underbrace{(A \cap B) \cup (A \cap C)}_{S_2}$$

$S_1 \subseteq S_2$

$S_2 \subseteq S_1$

$S_1 = A \cap (B \cup C) = \{x \in U \mid x \in A \land x \in (B \cup C)\}$

$(x \in B \lor x \in C)$

$\forall x \in S_1$

$\Leftrightarrow (x \in A) \land (x \in B \lor x \in C)$

$= (x \in A \land x \in B) \lor (x \in A \land x \in C)$

$(x \in A \cap B) \lor (x \in A \cap C)$

$x \in$

Proving set identities

$X = Y$

- All of these set identities claim that some set X is equal to some set Y.
- Remember: two sets X and Y are equal if and only if $X \subseteq Y$ and $Y \subseteq X$.
- So, given sets X and Y, to prove that $X = Y$:

1. Prove that $X \subseteq Y$. ✓
2. Prove that $Y \subseteq X$. ✓

Example proof - set identity

- For all sets A, B and C,

  ① $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

  [This is one of the distributive laws for sets. The book proves the other one, so we'll prove this one.]

- **Proof**: (on the chalkboard).

- For all sets A and B, $(A \cup B)^c = A^c \cap B^c$.

  [This is one of DeMorgan's Laws for sets.]

- **Proof**: (on the chalkboard).

- For all sets A and B, $(A - B) \cup (A \cap B) = A$.

- **Proof**: (on the chalkboard).

① $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ ✓

$\forall x \in U.\ x \in A \cap (B \cup C)$

$\equiv (x \in A) \wedge (x \in B \cup C)$

$\equiv (x \in A) \wedge (x \in B \vee x \in C)$

$\equiv (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$

$\equiv (x \in A \cap B) \vee (x \in A \cap C)$

$\equiv x \in (A \cap B) \cup (A \cap C)$

② $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

$\therefore \forall x \in U.\ x \in (A \cap B) \cup (A \cap C)$

$\equiv x \in (A \cap B) \vee x \in (A \cap C)$

$\equiv (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$

$\equiv (x \in A) \wedge (x \in B \vee x \in C)$

$\equiv (x \in A) \wedge x \in (B \cup C)$

$\equiv x \in (A \cap (B \cup C))$

Example proof - set identity

D $(A\cup B)^c = A^c\cap B^c$

$\forall x\in U.\ x\in (A\cup B)^c$

$\equiv \neg\neg(\forall x\in U.\ x\in(A\cup B)^c)$

$\equiv \neg(\exists x\in U.\ \neg(x\in(A\cup B)^c))$

$\equiv \neg(\exists x\in U.\ x\notin(A\cup B)^c)$

$\equiv \neg(\exists x\in U.\ x\in A\cup B)$

$\equiv \neg(\exists x\in U.\ x\in A \vee x\in B)$

$\equiv \forall x\in U.\ \neg(x\in A \vee x\in B)$

$\equiv \forall x\in U.\ \neg(x\in A) \wedge \neg(x\in B)$

$\equiv \forall x\in U.\ x\notin A \wedge x\notin B$

$\equiv \forall x\in U.\ x\in A^c \wedge x\in B^c$

$\equiv \forall x\in U.\ x\in A^c\cap B^c$

- For all sets A, B and C,

$$A\cap(B\cup C) = (A\cap B)\cup(A\cap C).$$

[This is one of the distributive laws for sets. The book proves the other one, so we'll prove this one.]

- **Proof**: (on the chalkboard).

- For all sets A and B, $(A\cup B)^c = A^c\cap B^c$.

[This is one of DeMorgan's Laws for sets.]

(2)

- **Proof**: (on the chalkboard).

- For all sets A and B, $(A-B)\cup(A\cap B) = A$.

- **Proof**: (on the chalkboard).

$\forall x\in U.\ x\in A^c\cap B^c$

$\equiv \neg\neg(\forall x\in U.\ x\in A^c\cap B^c)$

$\equiv \neg(\exists x\in U.\ \neg(x\in A^c\cap B^c))$

$\equiv \neg(\exists x\in U.\ \neg(x\in A^c\wedge x\in B^c))$

$\equiv \neg(\exists x\in U.\ x\in A\vee x\in B)$

$\equiv \neg(\exists x\in U.\ x\in A\cup B)$

$\equiv \forall x\in U.\ \neg(x\in A\cup B)$

$\equiv \forall x\in U.\ x\in(A\cup B)^c$

Example proof - set identity

$A = B \iff A \subseteq B \land B \subseteq A$

- For all sets A, B and C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

[This is one of the distributive laws for sets. The book proves the other one, so we'll prove this one.]

- **Proof**: (on the chalkboard).

- For all sets A and B, $(A \cup B)^c = A^c \cap B^c$.

[This is one of DeMorgan's Laws for sets.]

- **Proof**: (on the chalkboard).

- For all sets A and B, $(A - B) \cup (A \cap B) = A$.

- **Proof**: (on the chalkboard).

$\forall x \in U.\ x \in (A-B) \cup (A \cap B)$

$\equiv x \in (A-B) \lor x \in (A \cap B)$

$\equiv (x \in A \land x \notin B) \cup (x \in A \land x \in B)$

$\equiv (x \in A) \land (x \notin B \lor x \in B)$

$\equiv (x \in A) \land (x \in B^c \lor x \in B)$

$\equiv (x \in A) \land (x \in (B^c \cup B))$

$\equiv (x \in A) \land (x \in U)$

$\equiv x \in (A \cap U)$

$\equiv x \in A.$

Empty set

$\phi \quad |\phi| = 0$

- There is only one set with no elements. (See book for proof.)
- **Proving that a set X is empty:**
  - Suppose that X has an element and derive a contradiction.
- **Example:**

  Let A and B be any subsets of some universal set U.

  Prove that $(A \cap B) \cap (A - B) = \emptyset$.
- **Proof**: (on the chalkboard).

$P: \boxed{(A \cap B) \cap (A - B) = \phi}$

negation $\boxed{\neg P: (A \cap B) \cap (A - B) \neq \phi}$

proof by contradiction.

Suppose $\neg P$ is true. $\therefore (A \cap B) \cap (A - B) \neq \phi$

$\exists x \in U$. $\boxed{x \in (A \cap B) \cap (A - B)} \neq \phi$

$\equiv x \in (A \cap B) \wedge x \in (A - B)$

$\equiv (x \in A \wedge x \in B) \wedge (x \in A \wedge x \notin B)$

$\equiv (x \in A \wedge x \in A) \wedge x \in B \wedge x \notin B$

$\equiv x \in A \wedge (x \in B \wedge x \in B^c)$

$\equiv x \in A \wedge (x \in B \wedge B^c)$

$\equiv x \in A \wedge (x \in \phi)$

$\therefore \neg P$ is false

$\therefore P$ is true.

For all sets A, B, and C, if $B \cap C \subseteq A$, then $(C - A) \cap (B - A) = \emptyset$.

P: $\forall A, B, C \quad B \cap C \subseteq A \rightarrow (C-A) \cap (B-A) = \phi$ ✓

negation $\neg P$: $\exists A, B, C, \quad (B \cap C \subseteq A) \wedge \neg((C-A) \cap (B-A) = \phi)$

$\exists A, B, C, \quad \boxed{(B \cap C \subseteq A)} \wedge (C-A) \cap (B-A) \neq \phi$

proof by contradiction : suppose $\neg P$ is true. $(C-A) \cap (B-A) \neq \phi$

$\exists x \in U, \quad x \in (C-A) \cap (B-A) \neq \phi$

$\equiv x \in (C-A) \wedge x \in (B-A)$

$\equiv (x \in C \wedge x \in A^c) \wedge (x \in B \wedge x \in A^c)$

$\equiv (x \in B) \wedge (x \in C) \wedge (x \in A^c)$

$\equiv (x \in B \cap C) \wedge (x \in A^c)$

$\therefore \boxed{x \in B \cap C} \quad \boxed{x \in A^c}$

$x \in B \cap C \rightarrow x \in A$

$x \in B \cap C$

$\therefore x \in A$

$x \in A^c$

$(x \in A) \wedge (x \in A^c) = c$

$\therefore \neg P$ is false

$\therefore P$ is true.

For all sets A, B, C, and D, if $A \cap C = \varnothing$, then $(A \times B) \cap (C \times D) = \varnothing$.

P: $\forall A, B, C, D \quad A \cap C = \phi \longrightarrow (A \times B) \cap (C \times D) = \phi$

negation $\boxed{\neg P:}$ $\exists A, B, C, D \quad \boxed{A \cap C = \phi} \wedge \left((A \times B) \cap (C \times D) \neq \phi \right)$

proof by contradiction. suppose $\neg P$ is true. $(A \times B) \cap (C \times D) \neq \phi$

$\exists (x, y) \quad (x, y) \in (A \times B) \cap (C \times D)$

$\equiv ((x, y) \in A \times B) \wedge ((x, y) \in C \times D)$

$\equiv (x \in A \wedge y \in B) \wedge (x \in C \wedge y \in D)$

$\equiv x \in A \wedge x \in C \wedge y \in B \wedge y \in D$

$\equiv (x \in A \cap C) \wedge (y \in B \cap D)$

$x \in A \cap C \Rightarrow A \cap C \neq \phi$, $A \cap C = \phi \Rightarrow C$

$\therefore \neg P$ is false

$\therefore P$ is true.